

# Odd multiperfect numbers

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## Abstract

A natural number  $n$  is called *multiperfect* or *k-perfect* for integer  $k \geq 2$  if  $\sigma(n) = kn$ , where  $\sigma(n)$  is the sum of the positive divisors of  $n$ . In this paper, we establish the structure theorem of odd multiperfect numbers analogous as Euler's theorem on odd perfect numbers. We prove the divisibility of the Euler part of odd multiperfect numbers and characterize the forms of odd perfect numbers  $n = \pi^\alpha M^2$  such that  $\pi \equiv \alpha \pmod{8}$ . We also present some examples to show the nonexistence of odd perfect numbers as applications.

*Key words:* Odd multiperfect numbers; Euler part; nonexistence

*MSC:* 11A05; 11A25

## 1. Introduction

Let  $n$  be a positive integer and  $\sigma(n)$  be the sum of the positive divisors of the natural number  $n$ . We say  $n$  is *multiperfect* or *k-perfect* if  $\sigma(n) = kn$ . For example, 6 is a 2-perfect number and 120 is a 3-perfect number. One can see [4] or [9] for a survey of multiperfect numbers.

If  $n$  is an even 2-perfect number  $n$ , then the well known Euclid-Euler theorem states that the only even 2-perfect number is  $2^{p-1}(2^p-1)$ , where  $2^p-1$  is Mersenne prime. The odd case is different. It is not known that whether odd  $k$ -perfect numbers exist for any  $k \geq 2$ . But some properties of odd  $k$ -perfect numbers have been investigated. For odd 2-perfect number  $n$ , Euler has shown that  $n$  has the form  $n = \pi^\alpha M^2$ , where  $\pi$  is prime,  $(\pi, \alpha) = 1$  and  $\pi \equiv \alpha \equiv 1 \pmod{4}$ .  $\pi^\alpha$  is called the Euler factor. Recently, Brogghan and Zhou extended Euler's theorem to 4-perfect numbers, and then to  $2^k$ -perfect numbers [1]. One goal of this paper is to obtain a slightly more precise description of the structure of odd  $k$ -perfect numbers. It turns out that an odd  $k$ -perfect number  $n$  has the form  $n = \Pi M^2$  for which  $(\Pi, M) = 1$  and all exponents of prime factors of  $\Pi$  are odd (see Theorem 2.2 for details).  $\Pi$  is called the *Euler part* of the multiperfect number  $n$  which is analogous as the Euler factor of 2-perfect numbers.

It is interesting to study the Euler part of  $k$ -perfect numbers. It turns out that

the properties of the Euler part of  $k$ -perfect numbers can be used to prove the nonexistence of odd  $k$ -perfect numbers. For instance, Starni [10] recently proved that if an odd 2-perfect number  $n$  has the form  $n = \pi^\alpha 3^{2\beta} Q^{2\beta}$  with  $(3, Q) = 1$ , then  $3^{2\beta} | \sigma(\pi^\alpha)$ . Using this result, he showed that if  $\pi \equiv 1 \pmod{12}$ ,  $\alpha \equiv 1, 9 \pmod{12}$ , then there does not exist odd 2-perfect numbers  $n = \pi^\alpha 3^{2\beta} M^2$ . This result was extended to odd  $2^k$ -perfect numbers by Brogghan and Zhou [1]. We will show that the prime 3 can be replaced with the Fermat prime under suitable conditions (see Corollary 3.2).

An early result of Starni [11] on the odd 2-perfect numbers is that there is no odd 2-perfect numbers decomposable into primes all of the type  $\equiv 1 \pmod{4}$  if  $n = \pi^\alpha M^2$  and  $\pi \not\equiv \alpha \pmod{8}$ . In this paper, we will characterize the forms of odd 2-perfect numbers such that  $\pi \equiv \alpha \pmod{8}$ . As a consequence, we extend Starni's results and show the nonexistence of some forms of odd perfect numbers.

## 2. Structure of multiperfect numbers

Recently, Broughan and Zhou [1] extended Euler's structure theorem of odd 2-perfect numbers to odd 4-perfect numbers, and then to odd  $2^k$ -perfect numbers. From the proof of the structure of odd 4-perfect number, they observed the following beautiful fact, which is of independent interest.

**Theorem** (Broughan and Zhou [1, Theorem 2.2]). *For all odd primes  $p$ , powers  $j \geq 1$  and odd exponents  $e > 0$ , we have*

$$2^j \| \sigma(p^e) \iff 2^{j+1} \| (p+1)(e+1).$$

This theorem was proved by discussing different cases and constructing some polynomials. Here we will give a short proof by an element argument. More general, we have the following

**Theorem 2.1.** *Let  $p$  be a prime and  $e$  be a positive integer. Let  $\nu_2(m)$  be the highest power of 2 dividing integer  $m$ . Then we have*

$$\nu_2(\sigma(p^e)) = \begin{cases} \nu_2(p+1) + \nu_2\left(\frac{e+1}{2}\right), & \text{if } p > 2 \text{ and } e \equiv 1 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

$$\nu_2(\sigma(p^e) - 1) = \begin{cases} 0, & \text{if } p > 2 \text{ and } e \equiv 1 \pmod{2}, \\ \nu_2(p+1) + \nu_2\left(\frac{e}{2}\right), & \text{if } p > 2 \text{ and } e \equiv 0 \pmod{2}, \\ 1, & \text{if } p = 2. \end{cases} \quad (2)$$

**Proof.** It is obvious that  $\sigma(p^e) = 1 + p + \cdots + p^e \equiv 1 \pmod{2}$  when  $p = 2$  or the case  $p$  is odd and  $e$  is even. Now we assume that  $p$  and  $e$  are both odd. We

write  $r = \nu_2\left(\frac{e+1}{2}\right)$  and  $\frac{e+1}{2} = 2^r s$  for some odd integer  $s$ . Then

$$\begin{aligned}
\sigma(p^e) &= \frac{p^{e+1} - 1}{p - 1} \\
&= \frac{(p^2)^{\frac{e+1}{2}} - 1}{p - 1} \\
&= \frac{(p^2)^{2^r s} - 1}{p - 1} \\
&= (p + 1) \frac{(p^{2s})^{2^r} - 1}{p^2 - 1} \\
&= (p + 1)((p^{2s})^{2^{r-1}} + 1)((p^{2s})^{2^{r-2}} + 1) \cdots (p^{2s} + 1) \frac{p^{2s} - 1}{p^2 - 1}.
\end{aligned}$$

Since  $p$  is odd, we have

$$(p^{2s})^{2^{r-i}} + 1 = (p^{s2^{r-i}})^2 + 1 \equiv 2 \pmod{4}, \quad 1 \leq i \leq r.$$

It follows that

$$\nu_2((p^{2s})^{2^{r-i}} + 1) = 1, \quad 1 \leq i \leq r.$$

Note that

$$\frac{p^{2s} - 1}{p^2 - 1} = 1 + p^2 + p^4 + \cdots + (p^2)^{s-1} \equiv s \equiv 1 \pmod{2}.$$

We get

$$\nu_2\left(\frac{p^{2s} - 1}{p^2 - 1}\right) = 0.$$

Therefore

$$\begin{aligned}
\nu_2(\sigma(p^e)) &= \nu_2(p + 1) + \sum_{i=1}^r \nu_2((p^{2s})^{2^{r-i}} + 1) + \nu_2\left(\frac{p^{2s} - 1}{p^2 - 1}\right) \\
&= \nu_2(p + 1) + r \\
&= \nu_2(p + 1) + \nu_2\left(\frac{e + 1}{2}\right).
\end{aligned}$$

The formula (2) follows from (1) and the fact

$$\nu_2(\sigma(p^e) - 1) = \nu_2(p\sigma(p^{e-1})) = \nu_2(p) + \nu_2(\sigma(p^{e-1})).$$

This complete the proof of Theorem 1.  $\square$

As an application of Theorem 2.1, we will establish the explicit structure theorem of  $k$ -perfect numbers for any integer  $k \geq 2$ .

**Theorem 2.2.** *Let  $n$  be odd and  $k$ -perfect with  $\nu_2(k) \geq 1$  and  $s$  be any integer satisfying  $1 \leq s \leq \nu_2(k)$ . Then  $n$  has the shape*

$$n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} M^2, \quad (3)$$

*where  $M$  is a positive integer, the  $p_i$  are primes with  $(p_i, M) = 1$  and the  $e_j$  are odd positive integers. If  $\nu_2(k) - s$  has a nonnegative partition*

$$\nu_2(k) - s = a_1 + a_2 + \cdots + a_s + b_1 + b_2 + \cdots + b_s, \quad a_i \geq 0, \quad b_j \geq 0, \quad (4)$$

*then the primes  $p_1, \dots, p_s$  satisfy*

$$p_i \equiv 2^{a_i+1} - 1 \pmod{2^{a_i+2}}$$

*and the exponents  $e_1, \dots, e_s$  satisfy*

$$e_j \equiv 2^{b_j+1} - 1 \pmod{2^{b_j+2}}.$$

Before proving the theorem, we give the definition of the Euler part of odd  $k$ -perfect numbers.

**Definition 2.3** *The Euler part of an odd  $k$ -perfect number  $n$  with the shape (3) is denoted by*

$$\Pi := p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}.$$

**Remark.**

(1) Theorem 2.2 shows that if  $n_1$  is  $k_1$ -perfect,  $n_2$  is  $k_2$ -perfect and  $\nu_2(k_1) = \nu_2(k_2)$ , then  $n_1$  and  $n_2$  have the same shapes. Therefore we only consider  $2^k$ -perfect numbers in many situations.

(2) Note that there are  $\nu_2(k)$  shapes of an odd  $k$ -perfect number  $n$  since  $s$  can take  $\nu_2(k)$  values and each  $s$  gives a shape of  $n$  as (3).

**Proof of Theorem 2.2.** Let  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$  where the  $p_i$  are distinct odd primes. Then  $n$  is  $k$ -perfect implies that

$$\sigma(n) = \sigma(p_1^{\alpha_1}) \sigma(p_2^{\alpha_2}) \cdots \sigma(p_r^{\alpha_r}) = kn.$$

Since  $\nu_2(k) \geq 1$ , it follows from (1) of Theorem 1 that some  $\alpha_i$  must be odd. Therefore we can write  $n$  as

$$n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \cdot q_1^{2f_1} q_2^{2f_2} \cdots q_t^{2f_t},$$

where  $e_1, \dots, e_s$  are odd positive integers,  $f_1, \dots, f_t$  are positive integers,  $p_1, \dots, p_s, q_1, \dots, q_t$  are odd primes. By Theorem 2.1 we have

$$\begin{aligned}
\nu_2(k) &= \nu_2(kn) \\
&= \nu_2(\sigma(n)) \\
&= \sum_{i=1}^s \nu_2(\sigma(p_i^{e_i})) + \sum_{j=1}^t \nu_2(\sigma(q_j^{2f_j})) \\
&= \sum_{i=1}^s \left( \nu_2(p_i + 1) + \nu_2\left(\frac{e_i + 1}{2}\right) \right) \\
&= s + \sum_{i=1}^s \left( \nu_2\left(\frac{p_i + 1}{2}\right) + \nu_2\left(\frac{e_i + 1}{2}\right) \right).
\end{aligned}$$

Therefore

$$\nu_2(k) - s = \sum_{i=1}^s \nu_2\left(\frac{p_i + 1}{2}\right) + \sum_{i=1}^s \nu_2\left(\frac{e_i + 1}{2}\right).$$

Given a nonnegative partition of  $\nu_2(k) - s$  such that

$$\nu_2(k) - s = a_1 + a_2 + \dots + a_s + b_1 + b_2 + \dots + b_s, \quad a_i \geq 0, \quad b_i \geq 0,$$

we take  $\nu_2\left(\frac{p_i + 1}{2}\right) = a_i$  and  $\nu_2\left(\frac{e_i + 1}{2}\right) = b_i, 1 \leq i \leq s$ . Note that

$$\begin{aligned}
\nu_2\left(\frac{c + 1}{2}\right) = d &\implies \frac{c + 1}{2} = 2^d(2l + 1) \text{ for some integer } l \geq 0 \\
&\implies c = 2^{d+2}l + 2^{d+1} - 1 \\
&\implies c \equiv 2^{d+1} - 1 \pmod{2^{d+2}}.
\end{aligned}$$

Theorem 2.2 follows immediately.  $\square$

We give two examples to recover the well known Euler's theorem on 2-perfect numbers and Broughan and Zhou's structure theorem on 4-perfect numbers [1, Theorem 2.1].

**Example 2.1.** Let  $n$  be an odd 2-perfect number. Then  $\nu_2(2) = 1, s = 1$ .  $n$  has the unique form  $n = \pi^\alpha M^2$  with  $\pi$  prime,  $\alpha$  odd and  $(\pi, M) = 1$ . By (4),  $a_1 = b_1 = 0$ . So  $\pi \equiv \alpha \equiv 1 \pmod{4}$ .

**Example 2.2.** Let  $n$  be an odd 4-perfect number. Then  $\nu_2(4) = 2$ .  $s = 1$  or  $2$ .

If  $s = 1$ , then  $n = p^e M^2$ . By (4),  $a_1 = 1, b_1 = 0$  or  $a_1 = 0, b_1 = 1$ . Therefore  $p \equiv 3 \pmod{8}, e \equiv 1 \pmod{4}$  or  $p \equiv 1 \pmod{4}, e \equiv 3 \pmod{8}$ .

If  $s = 2$ , then  $n = p_1^{e_1} p_2^{e_2} M^2$ . By (4),  $a_1 = a_2 = b_1 = b_2 = 0$ . Therefore  $p_1 \equiv p_2 \equiv e_1 \equiv e_2 \equiv 1 \pmod{4}$ .

### 3. Euler part of $2^k$ -perfect numbers

Based on a result of McDaniel [7], Starni [10] proved that if an odd 2-perfect number  $n$  has the form  $n = \pi^\alpha 3^{2\beta} Q^{2\beta}$  with  $(3, Q) = 1$ , then  $3^{2\beta} | \sigma(\pi^\alpha)$ . This result was generalized to odd  $2^k$ -perfect numbers by Broughan and Zhou [2, Theorem 2.6]. Recalling that  $\Pi$  is the Euler part of an odd  $k$ -perfect number. In the following Theorem 3.1 we will prove the divisible result of  $\sigma(\Pi)$  for a prime power. As a corollary, we extend the results mentioned above.

**Theorem 3.1.** *Let  $n = \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i}$  be an odd  $2^k$ -perfect number, where  $\Pi$  is the Euler part of  $n$ ,  $q$  and the  $p_i$  are distinct odd primes. Then*

$$q^{2\beta} | \sigma(\Pi) \iff \begin{cases} 2\beta_i + 1 \not\equiv 0 \pmod{\text{ord}_q(p_i)}, & \text{if } p_i \not\equiv 1 \pmod{q}, \\ 2\beta_i + 1 \not\equiv 0 \pmod{q}, & \text{if } p_i \equiv 1 \pmod{q}, \end{cases}$$

where  $\text{ord}_q(m)$  is the order of  $m$  in the multiplicative group  $(\mathbb{Z}/q\mathbb{Z})^*$ .

**Proof.** By the definition of  $2^k$ -perfect,

$$2^k n = 2^k \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i} = \sigma(\Pi) \sigma(q^{2\beta}) \sigma\left(\prod_{i=1}^s p_i^{2\beta_i}\right) = \sigma(n).$$

Therefore

$$q^{2\beta} | \sigma(\Pi) \iff (q, \prod_{i=1}^s \sigma(p_i^{2\beta_i})) = 1 \iff (q, \sigma(p_i^{2\beta_i})) = 1, \quad i = 1, \dots, s.$$

If  $p_i \equiv 1 \pmod{q}$ , then

$$\sigma(p_i^{2\beta_i}) \equiv 2\beta_i + 1 \pmod{q}.$$

Therefore

$$(q, \sigma(p_i^{2\beta_i})) = 1 \iff 2\beta_i + 1 \not\equiv 0 \pmod{q}.$$

If  $p_i \not\equiv 1 \pmod{q}$ , then

$$\begin{aligned} (q, \sigma(p_i^{2\beta_i})) = 1 &\iff \left(q, \frac{p_i^{2\beta_i+1} - 1}{p_i - 1}\right) = 1 \\ &\iff p_i^{2\beta_i+1} \not\equiv 1 \pmod{q} \\ &\iff 2\beta_i + 1 \not\equiv 0 \pmod{\text{ord}_q(p_i)}. \end{aligned}$$

The theorem follows.  $\square$

**Corollary 3.2.** *Let  $n = \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i}$  be an odd  $2^k$ -perfect as in Theorem 3.2. If  $q$  is a Fermat prime, that is  $q = 2^{2^t} + 1$  for some integer  $t \geq 0$ , and  $\prod_{i=1}^s (2\beta_i + 1) \not\equiv 0 \pmod{q}$ , then  $q^{2\beta} | \sigma(\Pi)$ .*

**Proof.** Note that the Euler function  $\phi(q) = 2^{2^t}$ . Lagrange's theorem implies that  $\text{ord}_q(p_i) | \phi(q)$ . This show that  $\text{ord}_q(p_i)$  is a power of 2. The corollary follows.  $\square$

**Remark.** If  $n = \pi^\alpha q^{2\beta} \prod_{i=1}^s p_i^{2\beta}$  be an odd 2-perfect number with  $\alpha \equiv \pi \equiv 1 \pmod{4}$ , then it is known that  $\beta \neq 2$  [6],  $\beta \neq 3$  [5],  $\beta \neq 5, 12, 17, 24, 62$  [8], and  $\beta \neq 6, 8, 11, 14, 18$  [2]. In [8] McDaniel and Hagis conjecture that there does not exist such 2-perfect numbers for any positive integer  $\beta$ . Corollary 3.2 can be used to prove this conjecture in some special cases. For example, if  $q = 5$ ,  $(2\beta + 1, 5) = 1$ ,  $\pi \equiv 1 \pmod{20}$ ,  $\alpha \equiv 13 \pmod{20}$ , then by Corollary 3.2,  $5 | \sigma(\pi^\alpha)$ , but  $\sigma(\pi^\alpha) \equiv 1 + \alpha \equiv 4 \pmod{5}$ . Therefore there does not exist such odd 2-perfect numbers. In particular,  $n = 41^{13} 5^{2\beta} \prod_{i=1}^s p_i^{2\beta}$  can not be an odd 2-perfect number for any  $\beta \not\equiv 2 \pmod{5}$  and odd primes  $p_i$ .

Let  $n = \pi^\alpha \prod_i p_i^{2\beta_i}$  be an odd 2-perfect number.  $\pi^\alpha$ , with  $\pi \equiv \alpha \equiv 1 \pmod{4}$ , is the Euler's factor. In [11] Starni proved the following results:

- (a)  $\pi \equiv \alpha \pmod{8}$  if each prime  $p_i \equiv 1 \pmod{4}$ .
- (b)  $\sigma(\pi^\alpha)/2$  cannot be prime if each prime  $p_i \equiv 3 \pmod{4}$ .

The conclusion (a) was proved based on a result of Ewell [3]. We will extend (a) and (b) in the following Theorem 3.3 and 3.4 respectively independent of Ewell's result. As a consequence, we obtain some results on the nonexistence of odd 2-perfect numbers.

**Theorem 3.3.** *Let  $n = \pi^\alpha M^2$  be an odd 2-perfect number, with  $\pi$  prime,  $(\pi, M) = 1$  and  $\pi \equiv \alpha \equiv 1 \pmod{4}$ . Then*

$$\sigma(M^2) \equiv 1 \pmod{4} \iff \pi \equiv \alpha \pmod{8}, \quad (5)$$

$$\sigma(M^2) \equiv 3 \pmod{4} \iff \pi \equiv \alpha + 4 \pmod{8}. \quad (6)$$

*In particular, if  $n = \pi^\alpha \prod_i p_i^{2\beta_i}$  with  $p_i \equiv 1 \pmod{4}$  or  $n = \pi^\alpha \prod_j q_j^{2\gamma_j}$  with  $q_j \equiv 3 \pmod{4}$ , then*

$$\pi \equiv \alpha \pmod{8}.$$

**Proof.** Using the fact that  $\sigma(\pi^\alpha M^2) = 2\pi^\alpha M^2$  and  $M, \alpha$  are odd, we find that

$$\pi^\alpha \equiv \pi \equiv \frac{\sigma(\pi^\alpha)}{2} \sigma(M^2) \pmod{8}. \quad (7)$$

Note that  $\pi \equiv \alpha \equiv 1 \pmod{4}$  implies  $\pi^4 \equiv \alpha^4 \equiv 1 \pmod{16}$ . It follows that

$$\begin{aligned} \sigma(\pi^\alpha) &= 1 + \pi + \cdots + \pi^\alpha \\ &= (1 + \pi + \pi^2 + \pi^3)(1 + \pi^4 + \pi^8 + \cdots + \pi^{\alpha-5}) + \pi^{\alpha-1}(1 + \pi) \\ &\equiv (1 + \pi)(1 + \pi^2) \frac{\alpha - 1}{4} + 1 + \pi \pmod{16}. \end{aligned}$$

Hence we have

$$\frac{\sigma(\pi^\alpha)}{2} \equiv (1 + \pi^2) \frac{1 + \pi}{2} \frac{\alpha - 1}{4} + \frac{1 + \pi}{2} \equiv \frac{1 + \pi}{2} \frac{1 + \alpha}{2} \pmod{8}.$$

It follows from (7) that

$$\pi \equiv \frac{\pi + 1}{2} \frac{\alpha + 1}{2} \sigma(M^2) \pmod{8}.$$

Since  $\frac{1+\pi}{2}$  is odd, we have

$$\frac{\pi(\pi + 1)}{2} \equiv \left( \frac{\pi + 1}{2} \right)^2 \frac{\alpha + 1}{2} \sigma(M^2) \equiv \frac{\alpha + 1}{2} \sigma(M^2) \pmod{8}. \quad (8)$$

If  $\sigma(M^2) \equiv 1 \pmod{8}$ , then  $\frac{\pi(\pi+1)}{2} \equiv \frac{\alpha+1}{2} \pmod{8}$  implies that

$$\pi(\pi + 1) \equiv \alpha + 1 \pmod{16}.$$

Recall that  $\pi \equiv \alpha \equiv 1 \pmod{4}$ . It is easy to find the solutions  $(\pi, \alpha) \pmod{16}$  are

$$(1, 1); \quad (5, 13); \quad (9, 9); \quad (13, 5).$$

In particular, we get

$$\pi \equiv \alpha \pmod{8}.$$

Similarly, by (8), one can find that

$$\begin{aligned} \sigma(M^2) \equiv 5 \pmod{8} &\implies \pi(\pi + 1) \equiv 5(\alpha + 1) \pmod{16} \\ &\implies (\pi, \alpha) \pmod{16} = (1, 9); (5, 5); (9, 1); (13, 13) \\ &\implies \pi \equiv \alpha \pmod{8}. \end{aligned}$$

$$\begin{aligned} \sigma(M^2) \equiv 3 \pmod{8} &\implies \pi(\pi + 1) \equiv 3(\alpha + 1) \pmod{16} \\ &\implies (\pi, \alpha) \pmod{16} = (1, 5); (5, 9); (9, 13); (13, 1) \\ &\implies \pi \equiv \alpha + 4 \pmod{8}. \end{aligned}$$

$$\begin{aligned} \sigma(M^2) \equiv 7 \pmod{8} &\implies \pi(\pi + 1) \equiv 7(\alpha + 1) \pmod{16} \\ &\implies (\pi, \alpha) \pmod{16} = (1, 13); (5, 1); (9, 5); (13, 9) \\ &\implies \pi \equiv \alpha + 4 \pmod{8}. \end{aligned}$$

This prove (5) and (6).

If we write  $M^2 = \prod_i p_i^{2\beta_i} \prod_j q_j^{2\gamma_j}$  where  $p_i \equiv 1 \pmod{4}$ ,  $q_j \equiv 3 \pmod{4}$ , then by (2) of Theorem 1  $\sigma(q_j^{2\gamma_j}) \equiv 1 \pmod{4}$ . Hence

$$\sigma(M^2) = \prod_i \sigma(p_i^{2\beta_i}) \prod_j \sigma(q_j^{2\gamma_j}) \equiv \prod_i (2\beta_i + 1) \pmod{4}. \quad (9)$$



Clearly, if  $\beta_i = 0$  for all  $i$ , that is  $M^2 = \prod_j q_j^{2\gamma_j}$ , then  $\sigma(M^2) \equiv 1 \pmod{4}$ , and (4) implies that  $\pi \equiv \alpha \pmod{8}$ .

If  $M^2 = \prod_i p_i^{2\beta_i}$  with  $p_i \equiv 1 \pmod{4}$ , then

$$2n = 2\pi^\alpha \prod_i p_i^{2\beta_i} = \sigma(\pi^\alpha) \sigma(M^2) = \sigma(n).$$

This shows that each prime factors of  $\sigma(M^2)$  is  $\equiv 1 \pmod{4}$ . Hence we have  $\sigma(M^2) \equiv 1 \pmod{4}$  and (5) implies that  $\pi \equiv \alpha \pmod{8}$ .  $\square$

**Remark.** By (6), (9) and (2) of Theorem 2.1, it is easy to see that  $\pi \equiv \alpha + 4 \pmod{8}$  if and only if the number of prime factors  $p^e$  of  $M^2$  with  $p^e \parallel M^2$ ,  $p \equiv 1 \pmod{4}$  and  $e \equiv 2 \pmod{4}$  is odd.

**Theorem 3.4** *Let  $n = \Pi M^2$  be an odd  $2^k$ -perfect number, where  $\Pi$  is the Euler part of  $n$ . If all prime factors of  $M$  are  $\equiv 3 \pmod{4}$  and  $\Pi = p_1^{e_1} \cdots p_s^{e_s} q_1^{f_1} \cdots q_{2t}^{f_{2t}}$  satisfies  $(\sigma(\Pi), p_1 \cdots p_s) = 1$ , where the primes  $p_i \equiv 1 \pmod{4}$ ,  $q_j \equiv 3 \pmod{4}$  and integer  $t \geq 0$ , then*

$$\Omega\left(\frac{\sigma(\Pi)}{2^k}\right) \equiv 0 \pmod{2},$$

where  $\Omega(\sigma(\Pi)/2^k)$  is the total number of prime factors of  $\sigma(\Pi)/2^k$ .

**Proof.** By the definition of  $2^k$ -perfect, we have

$$p_1^{e_1} \cdots p_s^{e_s} q_1^{f_1} \cdots q_{2t}^{f_{2t}} M^2 = \frac{\sigma(\Pi)}{2^k} \sigma(M^2).$$

Since  $(\sigma(\Pi), p_1 \cdots p_s) = 1$ , we deduce that all prime factors of  $\frac{\sigma(\Pi)}{2^k}$  are  $\equiv 3 \pmod{4}$ . Note that  $\sigma(M^2) \equiv 1 \pmod{4}$ . It follows that

$$1 \equiv p_1^{e_1} \cdots p_s^{e_s} q_1^{f_1} \cdots q_{2t}^{f_{2t}} M^2 = \frac{\sigma(\Pi)}{2^k} \sigma(M^2) \equiv (-1)^{\Omega(\frac{\sigma(\Pi)}{2^k})} \pmod{4}.$$

Therefore

$$\Omega\left(\frac{\sigma(\Pi)}{2^k}\right) \equiv 0 \pmod{2}. \quad \square$$

If  $n = \pi^\alpha M^2$  is an odd 2-perfect number such that all prime factors of  $M$  are  $\equiv 3 \pmod{4}$ , then Theorem 3.5 implies that

$$\Omega\left(\frac{\sigma(\pi^\alpha)}{2}\right) \equiv 0 \pmod{2}.$$

In particular, this show that  $\frac{\sigma(\pi^\alpha)}{2}$  can not be a prime. We can use this fact to prove the nonexistence of 2-perfect numbers. For example,  $\pi = 209$ ,  $\sigma(\pi) = 210 = 2 \cdot 3 \cdot 5 \cdot 7$  and  $\Omega(\frac{\sigma(\pi)}{2}) = 3$ .  $\pi = 30029$ ,  $\sigma(30029) = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

and  $\Omega(\frac{\sigma(\pi)}{2}) = 5$ . It follows that there does not exist 2-perfect number  $n = 209M^2$  and  $n = 30029M^2$  with all prime factors of  $M$  are  $\equiv 3 \pmod{4}$ .

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